ON THREE-DIMENSIONAL BODIES OF MINIMUM DRAG AT HIGH SUPERSONIC SPEED

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In ballistics and gas dynamics one finds the shape of the body of revolution having minimum drag for given length, volume, or other supplementary conditions.

From these solutions and from experiments it is known that use of a body of optimal form at hypersonic speeds permits a reduction of its wave drag (compared with the equivalent cone) by approximately 30 to 40 per cent (see [1] for example).

We here attempt to formulate and solve the problem of the form of the three-dimensional optimal body in hypersonic gas flow.

We consider flow past a body (Fig. 1) in a cylindrical system of coordinates ρ, φ, z with the z-axis chosen in the streamwise direction. We suppose that the surface of the body is described by the equation

$$\rho = r(\varphi) f(z) \qquad (1)$$

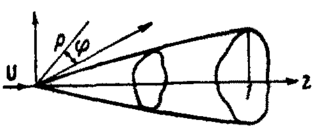

Fig. 1.

Taking the length of the body equal to unity, and the function $f(z)$ to be dimensionless, we can take $f(1) = 1$.

We shall determine the pressure on the surface of the body according to the Newtonian law, which can be written in the form

$$C_p = k \cos^2 (\mathbf{n}, \mathbf{U}) \left(C_p = \frac{2(p - p_0)}{\rho U^2} \right) \qquad (2)$$

where C_p is the pressure coefficient, k the coefficient of proportionality, **n** is the normal vector, and **U** the velocity vector of the oncoming stream. Using equations (1) and (2) it can be shown that the drag coefficient of the body, referred to the maximum area S is (for $f'(z) \geqslant 0$) given by the expression

$$C_x = \frac{K}{S_1} \int_0^{2\pi} r^4(\varphi)\, d\varphi \int_0^1 \frac{f(z) f'^3(z)\, dz}{1 + r'^2/r^2 + r^2 f'^2(z)} \tag{3}$$

Consideration is now limited to slender bodies; then the quantity $r^2 f'^2(z) \ll 1$ and it can be neglected in equation (3). As a result we have

$$C_x = \frac{K}{S_1} \int_0^1 f(z) f'^3(z)\, dz \int_0^{2\pi} \frac{r^4(\varphi)\, d\varphi}{1 + r'^2/r^2} \tag{4}$$

Representation of the drag in the form (4) allows one to reduce the problem of finding the surface of the optimal three-dimensional body to the separate problems of the determination of the optimal forms of the meridional curve and of the cross-section. The shape of the optimal meridian curve was found previously within the Newtonian approximation, and is well known to have the form $f \sim z^{3/4}$. To find the cross-sectional contour we formulate the following variational problem. In the class of smooth curves $r(\varphi)$ with a finite number of discontinuities in the first derivative to find the minimum of the functional

$$J = \int_0^{2\pi} \frac{r^4(\varphi)\, d\varphi}{1 + r'^2/r^2} \qquad \left(S_1 = \frac{1}{2} \int_0^{2\pi} r^2(\varphi)\, d\varphi\right)$$

for given maximum cross-section S_1 and a given characteristic dimension r^0. As is well known, the extremals for a variational problem of this kind must satisfy the Euler equation for the function

$$F = \frac{r^4}{1 + r'^2/r^2} + \lambda^* r^2$$

Also, along the extremals the Legendre condition $F_{r'r'} \geqslant 0$ must be fulfilled, and at points of discontinuity of the derivative the Weierstrass-Erdmann conditions

$$[F - r'F_{r'}]_{\varphi-0} = [F - r'F_{r'}]_{\varphi+0}, \qquad [F_{r'}]_{\varphi-0} = [F_{r'}]_{\varphi+0} \tag{5}$$

Because the expression for F does not contain the independent variable explicitly, the Euler equation admits the integral

$$r_1^4\left(1+3\frac{r_1'^2}{r_1^2}\right)=(\pm 1-\lambda r_1^2)\left(1+\frac{r_1'^2}{r_1^2}\right)^2 \qquad \left(\begin{array}{l} r_1=r\,(|C|)^{1/4} \\ \lambda=\lambda^*\,(|C|)^{1/2}\end{array}\right) \tag{6}$$

Here C is a constant of integration. Equation (6) consists of two irreducible fourth-degree differential equations and contains, generally speaking, eight families of integral curves. Without dwelling on the details of the qualitative investigation of this equation, we note that an extremal with the above properties is not included entirely in any one family of integral curves, and that four of the eight families of integral curves cannot be extremals in any part. We now transform equation (6) into parametric form by means of the substitution $r_1^2/r_1'^2 = t^2$ (where the subscript 1 is dropped).

The remaining four families can be represented in the form

$$r^2=\frac{(-1)^i\,2}{\lambda\,(1+\sqrt{1+(-1)^i q\,(t,\lambda)})}\quad (i=0,1),\qquad d\varphi=\pm\frac{1}{2}t\frac{dr^2}{r^2},\quad q\,(t,\lambda)=\frac{4t^2\,(3+t^2)}{\lambda^2\,(1+t^2)^2}. \tag{7}$$

From equation (7) it follows immediately that if $i = 0$ then $\lambda > 0$, and if $i = 1$ then $\lambda < 0$. By virtue of the Legendre condition the parameter t is contained within the limits 0 to $\sqrt{3}$.

Integrating relation (7) for $\lambda > 0$, we find that in parametric form the solution is given by

$$r^2=\frac{2}{\lambda\,(1+\sqrt{1+q})},\qquad \varphi=\frac{2}{\lambda^2}\int_0^t\frac{(3-t^2)\,t^2dt}{(1+\sqrt{1+q})\,\sqrt{1+q}\,(1+t^2)^3} \tag{8}$$

If $\lambda < 0$, then we have correspondingly

$$r^2=\frac{2}{\lambda\,(1+\sqrt{1-q})},\qquad \varphi=\frac{2}{\lambda^2}\int_0^t\frac{(3-t^2)\,t^2dt}{(1+\sqrt{1-q})\,\sqrt{1-q}\,(1+t^2)^3} \tag{9}$$

From equations (8) and (9) only two families of integral curves are determined. The remaining two families are obtained with a minus sign in the second of equations (7).

If we form an estimate for the magnitude of the angle φ, it turns out that its greatest possible value is less than $\pi/8$.

Consequently there are no closed curves in the families of integrals, and all the curves are arranged in a certain angle $0 \leqslant \varphi \leqslant \varphi_0 = \varphi(t_0)$.

We now consider the third family of integral curves that is obtained with $\lambda > 0$ and a minus sign in the second of equations (7). It is easy to show that by reducing the parameter t from the value t_0 to zero we obtain curves of the third family, situated in the angle $\varphi_0 \leqslant \varphi \leqslant 2\varphi_2$. Joining curves of the first and third family, we can obtain integral curves in the range $0 \leqslant \varphi \leqslant 2\varphi_0$, where the continuations of the lines of the first family are curves symmetric with respect to the original ones in the plane $\varphi = \varphi_0$. To further extend the extremals to greater angles one again uses the first family of integral curves and repeats the previous construction. Carrying out a finite number of such constructions, it is possible to construct extremals filling an angle of arbitrary size. If we thereby require that the angle φ_0 be a submultiple of 2π the extremal is closed. It is readily observed that at points of joining ($t = 0$ and $t = t_0$) of the pieces of which the extremal is formed the derivative suffers a discontinuity, so that generally speaking conditions (5) cannot be satisfied at these points. Checking shows, however, that the first of conditions (5) is satisfied identically at all junction points, but the second only at points where $t = 0$, that is, at the maximum value of the radius.

We shall now take as the characteristic linear dimension of the body the minimum value of the radius. The variation of the radius at points where $t = t_0$ is equal to zero and the second of conditions (5) drops out. Similarly, it can be shown that the solution represented by equation (9) corresponds to the case when the characteristic dimension is taken equal to the maximum value of the radius. We derive relations for determining the constants c, t_0 and λ. By virtue of the closure of the extremal we have the equality

$$\frac{2}{\lambda^2}\int_0^{t_0}\frac{(3-t^2)\,t^2dt}{(1+\sqrt{1+q})\sqrt{1+q}\,(1+t^2)^3} = \frac{\pi}{n} \tag{10}$$

where n is a positive integer determining the number of double pieces of which the extremal is composed. Turning to the isoperimetric condition and using equation (8) we find

$$\frac{2}{\lambda^2}\int_0^{t_0}\frac{(3-t^2)\,t^2dt}{(1+\sqrt{1+q})^2\sqrt{1+q}\,(1+t^2)^3} = \frac{S_1}{nr^{\circ 2}(1+\sqrt{1+q_0})} \qquad (q_0 = q\,(t_0,\lambda)) \tag{11}$$

Recalling now that $r_{\min} = r^{\circ}$, we find a third condition in the form

$$2\sqrt{|c|} = r^{\circ 2}\lambda\,(1+\sqrt{1+q_0}) \tag{12}$$

Investigation of relations (10) to (12) shows that they can be satisfied if n is sufficiently great. In particular, the parameters λ and t_0

are found for fixed n from conditions (10) and (11), and the parameter c from condition (12). Thus it is possible to obtain a countable set of extremals satisfying all the conditions of the variational problem. Thus the form of the cross-section is entirely determined by the parameters n and $S_1/r^{\circ 2}$, the quantity r° characterizing only the scale of the body.

The set of extremals obtained may for simplicity be regarded as analogous to the set of extremum points of a function such as $y = ax + \sin x (0 < a < 1)$. We now consider the drag of the body obtained. From equation (4) we have

$$C_x \Big/ \frac{K}{S_1}\int_0^1 ff'^3 dz = \frac{4nr^{\circ 4}}{\lambda^2}(1+\sqrt{1+q_0})^2 \int^{t_0} \frac{(3-t^2)\,t^4 dt}{(1+\sqrt{1+q})^3\sqrt{1+q}\,(1+t^2)^4} \tag{13}$$

The solution of the variational problem given by the above equations is rather complicated for practical calculation. We therefore study it in greater detail for values of the parameter $t_0 \ll 1$, since in this case the solution is considerably simplified.

After some calculations from equations (8) to (13) we obtain

$$r^2 = 2/\lambda(1+\sqrt{1+z^2}) + O(t^4), \qquad \varphi = \frac{\lambda}{4\sqrt{3}}\int_0^z \frac{z^2 dz}{(1+\sqrt{1+z^2})\sqrt{1+z^2}} + O(t^5)$$

$$\left(z = \frac{t\sqrt{12}}{\lambda}\right)$$

$$\frac{\lambda}{4\sqrt{3}}\int_0^{z_0} \frac{z^2 dz}{(1+\sqrt{1+z^2})\sqrt{1+z^2}} + O(t^5) = \frac{\pi}{n} \qquad \left(z_0 = \frac{t_0\sqrt{12}}{\lambda}\right)$$

$$\frac{\lambda}{4\sqrt{3}}\int_0^{z_0} \frac{z^2 dz}{(1+\sqrt{1+z^2})^2\sqrt{1+z^2}} + O(t^5) = \frac{S_1}{nr^{\circ 2}(1+\sqrt{1+z_0^2})} \tag{14}$$

$$C_x \Big/ \frac{K}{S_1}\int_0^1 ff'^3 dz = \frac{nr^{\circ 4}\lambda^3}{6\sqrt{3}}(1+\sqrt{1+z_0^2})^2 \int_0^{z_0} \frac{z^4 dz}{(1+\sqrt{1+z^2})^3\sqrt{1+z^2}} + O(t^7)$$

where all integrals appearing in (14) are calculated in terms of elementary functions. As a result the equation of the sections of which the cross-section of the body is composed takes the form

$$r^2 = \frac{r^{\circ 2}(1+\sqrt{1+z_0^2})}{1+\sqrt{1+z^2}}, \qquad \varphi = \frac{\lambda}{4\sqrt{3}}[z - \ln(z+\sqrt{1+z^2})] \qquad 0 \leqslant z \leqslant z_0 \tag{15}$$

The parameters of this curve are determined from the conditions

$$\frac{\lambda}{4\sqrt{3}}[z_0 - \ln(z_0 + \sqrt{1+z_0^2})] = \frac{\pi}{n}$$
$$\frac{\lambda}{4\sqrt{3}}\left[\ln\left|\frac{1+z_0+\sqrt{1+z_0^2}}{1-z_0+\sqrt{1+z_0^2}}\right| - \frac{2z_0}{1+\sqrt{1+z_0^2}}\right] = \frac{S_1}{nr^{o2}(1+\sqrt{1+z_0^2})} \tag{16}$$

We note that equations (15) are written in the original coordinates.

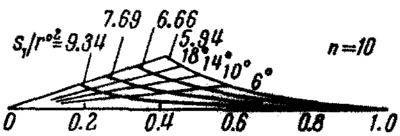

Fig. 2.

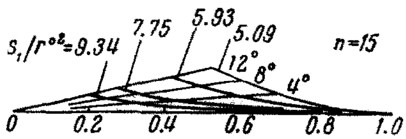

Fig. 3.

The drag coefficient for the case under consideration can be written as

$$C_x \Big/ \frac{K}{S_1}\int_0^1 ff'^3 dz = \frac{nr^{o4}\lambda^3}{6\sqrt{3}}(1+\sqrt{1+z_0^2})^2\left[\frac{4z_0}{1+\sqrt{1+z_0^2}} - \right.$$
$$\left. -3\ln\left|\frac{1+z_0+\sqrt{1+z_0^2}}{1-z_0+\sqrt{1+z_0^2}}\right| + \frac{2z_0(1+\sqrt{1+z_0^2})}{(1+\sqrt{1+z_0^2})^2 - z_0^2}\right] \tag{17}$$

Calculation of the cross-section of the body from these equations offers no difficulty, and results are shown in Figs. 2 and 3 for a number of values of the parameter S_1/r^{o2} with $n = 10$ and $n = 15$. From these graphs it follows that the walls of the body are stretched along the radius and have a ratio $1.9 \lesssim r_{max}/r_{min} \lesssim 5$. As a result of the cross-section of the body has a star-shaped form with very pronounced points. Figure 4 shows the general shape of the cross-section of the body with $n = 10$ and $S_1/r^{o2} = 5.94$ and 9.34. Calculation of the drag of these bodies according to equation (17) shows that it is less than the drag of the equivalent optimal body of revolution (having the same length and maximum area) by, for example, a factor of twenty. Using (13) it is not difficult to see that the drag decreases with increase of the number of points n, and tends to zero in the limit $n \to \infty$. Figure 5 shows the variation of the drag coefficient C_x, referred to the value $C_x^{\,0}$ for the equivalent optimal body of revolution according to Newton, with the parameter S_1/r^{o2} for different values of the number of points n.

As is evident from the graph, with increase in the number n the curves lie lower, so that the drag is decreased, as it is also when the parameter S_1/r^{o2} increases.

From these graphs one may conclude that the drag of optimal bodies for which the variational problem has a solution is less than the drag of the equivalent optimal bodies of revolution by a factor of twenty or

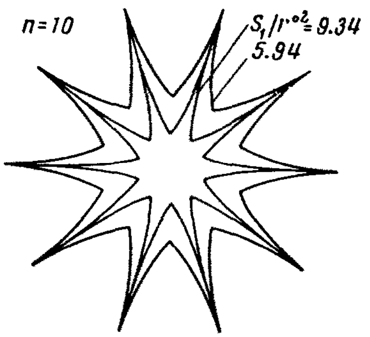

Fig. 4.

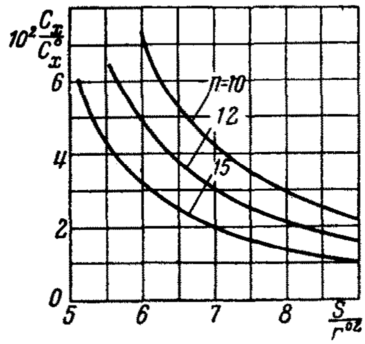

Fig. 5.

more. This result is obviously exaggerated. The fact is that the surface of the optimal body obtained is ridged, and the Newtonian flow scheme is not justified in the vicinity of a ridge, so that in a more precise formulation such strong reductions of drag would not be observed. Naturally the drag of the body would actually remain finite for $n \to \infty$, and an absolute optimum would evidently be obtained with a finite number of points. The practical utility of optimal bodies with a large number of points is also less effective because of the presence of the boundary layer.

We turn our attention now to a property associated with the shape found for the cross-section of optimal bodies. According to the Newtonian flow scheme, the drag of a body is not changed if the lobes of which it is composed are rearranged in a different order. Consequently there appears the possibility of creating from one shape a set of others. These shapes will have from a mathematical point of view points of discontinuity, and from a physical one unnecessary friction surfaces. However, the results obtained above allow one to solve without difficulty one variational problem, consisting in the determination of the optimal shape of a body one of whose lateral surfaces is a plane parallel to the stream. The solution follows immediately from the principle of symmetry, and the shape of the cross-section for $n = 10$ is shown in Fig. 6.

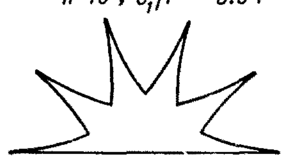

Fig. 6.

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